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Effect of Tip Vortex Geometry on the Flow Through the Blades of Hovering Rotors

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Introduction

IT is well known that the average velocity through the blades of a hovering rotor, U_b, is greater than half the average velocity in the far wake, U_w, the value required by simple momentum theory. This is reflected in the ratio of the radius of the far wake, r_w, to the blade tip radius r_b, which has an experimental value of around 0.78, rather than $2^{-1/2}$ (e.g., see Leishman[1]). The purpose of this Note is to investigate the role of wake geometry, in the form of the pitch and radius of the tip vortices, in producing the discrepancy. It is suggested that the discrepancy is due mainly to the change in pitch of the tip vortices and is reduced by their contraction. The basic geometry of the rotor and the tip vortex is shown Fig. 1 for a single-blade rotor.

Formulation and Analytical Results

It is necessary to establish the following relation between the circulation of each tip vortex, Γ, and U_w:

$$U_w = N\Gamma/(2\pi k_2) \tag{1}$$

where N is the number of blades and k_2 is the pitch of the tip vortices in the far wake. To do this, consider the circulation around

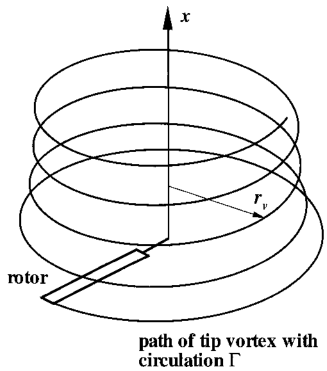

Fig. 1 Geometry of the tip vortex of a single-bladed rotor; angular velocity of the blade is Ω rad/s.

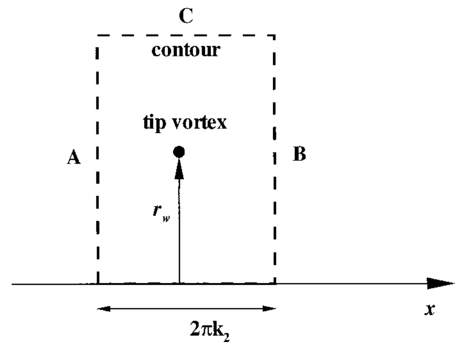

Fig. 2 Rectangular circuit used to derive Eq. (1).

the rectangular circuit shown in Fig. 2 with one of the two legs of length $2\pi k_2$ placed along the axis of rotation ($r = 0$). The radial legs (A and B in Fig. 2) are greater than r_w, to enable the circuit to enclose N tip vortices without cutting them. Thus, the area integral of vorticity within the circuit is $N\Gamma$. Because the contribution to the circulation around the circuit from the two radial legs will cancel and that of the outer leg C will be zero, Eq. (1) gives the velocity along the axis of rotation. If the radial leg of the contour is now reduced to be less than r_w, then the circulation becomes zero, which requires Eq. (1) to be independent of radius in the far wake.

Relation (1) is strictly valid only if all of the tip vorticity is confined within tip vortices of constant pitch and radius and the hub vorticity is confined to N straight vortex lines lying along the axis of rotation. This is a simplification of the structure of real wakes and may well be the reason for the overestimated value for Γ that can be determined from Eq. (1) as follows. The single rotor data of Leishman et al.[2] indicate that the velocity in the wake is reasonably uniform with r. When normalized by r_b and Ω, the angular velocity of the blades, the velocity was 0.073 at the last measuring position, $x/r_b = 0.399$ (see their Fig. 7), where the wake had not fully contracted. Their formula (2) for the development of the tip vortex radius allows U_w to be estimated as 0.0783. Using $k_2 = 0.041$ from their Table 2 gives $\Gamma/\Omega r_b^2 = 0.020$, which compares with the measured values in the range of 0.010–0.015 from their Fig. 13. Nevertheless, the simplicity of Eq. (1) allows the qualitative assessment of the effects of the tip vortex geometry.

To a similar level of simplification as that underlying Eq. (1), U_b is determined solely by the tip vortices. If the Biot–Savart law is written for $r = 0$ and the variation of U_b with r is ignored, then

$$U_b = \frac{N\Gamma}{4\pi}\int_0^\infty \frac{r_v^2}{k\left(r_v^2 + x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x \tag{2}$$

where r_v and k are the radius and pitch, respectively, of the tip vortices. The former is shown in Fig. 1 along with the coordinate x, which originates at the blades. Evaluating the Biot–Savart law

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only at $r = 0$ excludes terms that depend on the vortex angle, which would prevent a simple treatment of the integral. Because the inflow velocity usually increases with r, for example, Fig. 4 of Ref. 2, Eq. (2) must underestimate U_b. Hovering rotor experiments have shown that the vortex pitch has only two values: $k = k_1$ for $x \le x_1$, where $x_1 = Nk_1/2\pi$, and $k = k_2$ otherwise. This allows rewriting Eq. (2) as

$$U_b = \frac{N\Gamma}{4\pi}\left[\frac{1}{k_1}\int_0^{x_1}\frac{r_v^2}{\left(r_v^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x + \frac{1}{k_2}\int_{x_1}^{\infty}\frac{r_v^2}{k\left(r_v^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x\right] \tag{3}$$

If there were no contraction, the integrals are evaluated easily to give, using Eq. (1),

$$U_b = \frac{U_w}{2}\left[1 + \frac{x_1}{\left(r_b^2+x_1^2\right)^{\frac{3}{2}}}\left\{\frac{k_2}{k_1} - 1\right\}\right] \tag{4}$$

Since $k_2 > k_1$, $k_2/k_1 - 1$ is positive, then $U_b/U_w > \frac{1}{2}$. If $k = k_2$ throughout the wake, then $U_b/U_w = \frac{1}{2}$. However, to the level of approximation of the vortex structure,

$$U_b/U_w = (r_w/r_b)^2 \tag{5}$$

so that the change in the pitch is significant. A simple assessment of the effects of contraction can be made by writing

$$U_b > \frac{U_w}{2}\int_0^{\infty}\frac{r_v^2}{\left(r_v^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x \tag{6}$$

using Eq. (1) and $k_2 > k_1$. If the integrand in Eq. (6) is expressed in terms of $\mathrm{d}[x(r_v^2 + x^2)^{-1/2}]/\mathrm{d}x$, then integration by parts gives

$$U_b > (U_w/2)(1 - \delta) \tag{7a}$$

where

$$\delta = -\frac{1}{2}\int_0^{\infty}\frac{x\,\mathrm{d}r_v^2/\mathrm{d}x}{\left(r_v^2+x^2\right)^{\frac{3}{2}}}\,\mathrm{d}x \tag{7b}$$

which must always be positive for a monotonically contracting wake. Result (7) suggests that wake contraction acts to reduce the difference between U_b/U_w and $\frac{1}{2}$.

Numerical Results

Despite its simplicity, it appears impossible to integrate Eq. (2) analytically when it is combined with the empirical equations for r_v, such as Eq. (2) in Ref. 2 and the pitch data from their Table 2. The integral was evaluated numerically, using this Eq. (1) to remove the dependence on Γ. The result was $U_b/\Omega r_b = 0.0424$. Using Eq. (3), this gives $r_w/r_b = (0.0424/0.0786)^{1/2} = 0.74$, which is approximately halfway between the momentum theory value and the experimental result, but is closer to the latter if it is remembered that the analysis underestimates U_b. From Eq. (3), which gives the effect of the pitch change without contraction, $r_w/r_b = 0.87$.

Conclusion

It is concluded that the change in pitch of the tip vortex wake contraction is responsible for the average velocity through a hovering rotor being greater than the value from simple momentum theory. The effect of wake contraction is to reduce this discrepancy.

Acknowledgment

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Divergence and the *p–k* Flutter Equation

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Nomenclature

A	=	pitch amplitude
a	=	distance from midchord to pitch axis, positive aft, normalized by semichord
b	=	semichord
$C_{M\alpha}$	=	pitching moment coefficient slope, $M_\alpha/(1/2\rho U^2 b^2)$
I	=	mass moment of inertia about pitch axis per unit span
K	=	torsion spring stiffness
k	=	reduced frequency, $\omega b/U$, in general or $\mathrm{Im}(p)/V$ in the example
M	=	aerodynamic moment about pitch axis per unit span
m	=	mass per unit span
p	=	differential operator, $(2b/U)(\mathrm{d}/\mathrm{d}t)$, in general or $(1/\omega_\alpha)(\mathrm{d}/\mathrm{d}t)$ in the example
r_α	=	radius of gyration about the pitch axis, normalized by semichord
t	=	time
U	=	true airspeed
V	=	reduced velocity, $U/(\omega_\alpha b)$
$\alpha(t)$	=	instantaneous pitch angle
μ	=	mass ratio, $m/(\pi\rho b^2)$
ρ	=	air density
ω_α	=	pitch frequency in a vacuum

Introduction

In principle the flutter equation should be able to predict divergence. In Ref. 1 the use of the k method, as implemented in the FAST flutter analysis system, to determine divergence speeds of forward swept wings is described. The p and $p–k$ methods were used in Refs. 2 and 3 to study divergence of restrained and unrestrained systems. In the $p–k$ method, the real part of the eigenvalue p indicates whether the motion grows or dies out, whereas the imaginary part indicates the oscillation frequency. The generalized aerodynamic forces depend on p; therefore, an iterative solution procedure is generally used. In this study, the simplest aeroelastic divergence problem is solved using three different forms of the $p–k$ flutter equation, namely, that of Hassig[4] in which the aerodynamic coefficients depend only on the imaginary part of p; that of Rodden, Harder, and Bellinger[5] which introduces a dependence on the real part of p by dividing the generalized aerodynamic force matrix into an aerodynamic stiffness matrix and an aerodynamic damping matrix; and a form of the $p–k$ flutter equation that is equivalent to the g method of Chen.[6] The latter method employs a first-order

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